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SLOW AREA-PRESERVING DIFFEOMORPHISMS OF THE TORUS

BY

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ABSTRACT

We construct area-preserving real analytic diffeomorphisms of the torus with unbounded growth sequences of arbitrarily slow growth.

Given a smooth compact manifold M, consider the group $\mathrm{Diff}(M)$ of diffeomorphisms of M. For every $f \in \mathrm{Diff}(M)$ we define the growth sequence of f:

$$\Gamma_n(f) = \max(\max_{x\in M} \|d_x f^n\|, \max_{x\in M} \|d_x f^{-n}\|), \quad n \in \mathbb{N},$$

where f^n is the n-th iteration of f, f^{-n} is the n-th iteration of f^{-1}, and $\|d_x f\|$ is the operator norm of the differential of f at the point $x \in M$. Conjugations of f in the group $\mathrm{Diff}(M)$ generate equivalent growth sequences:

$$c(g)\Gamma_n(g^{-1}fg) \le \Gamma_n(f) \le C(g)\Gamma_n(g^{-1}fg), \quad g \in \mathrm{Diff}(M), n \in \mathbb{N}.$$

The asymptotics of the growth sequence is a basic dynamic invariant (see [3]). D'Ambra and Gromov [1, 7.10.C] proposed to study the behavior of growth sequences for various classes of diffeomorphisms. In particular, it is interesting to find examples of unbounded growth sequences of slow growth (see also the references in [1, 7.10.C]). We call the diffeomorphisms generating such growth sequences the slow diffeomorphisms.

Recently, Polterovich and Sodin [5] obtained several results on the growth sequences of smooth order-preserving diffeomorphisms of the interval $[0,1]$. In particular, they proved [5, Theorem 1.7] that for any sequence $\{a_n\}$ of positive

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numbers tending to infinity, there exists a C^∞-smooth diffeomorphism f, $f \neq \mathrm{Id}$, such that

$$\liminf_{n\to\infty} \frac{\Gamma_n(f)}{a_n} \leq 1.$$

On the other hand, a simple argument (see [5]) shows that for any diffeomorphism $f \neq \mathrm{Id}$,

$$\sum_{n\geq 1} \frac{1}{\Gamma_n(f)} < \infty.$$

Furthermore, Polterovich proved [4, Theorem 1.3] that for every $0 < \beta < 1$, there exists an area-preserving real analytic diffeomorphism f of the torus such that

$$\Gamma_n(f) \leq C n^\beta \log n,$$
$$\limsup_{n\to\infty} \frac{\Gamma_n(f)}{n^\beta} > 0.$$

In this note, we improve somewhat the result of Polterovich by producing area-preserving real analytic diffeomorphisms of the torus with arbitrarily slowly growing unbounded growth sequences.

THEOREM: *Let φ be a positive increasing (unbounded) function on $\mathbb{R}_+$ such that $\varphi(x) = o(x)$, $x \to \infty$. There exists an area-preserving real analytic diffeomorphism f of the torus such that*

$$\begin{cases} \Gamma_n(f) \leq \varphi(n), \\ \limsup_{n\to\infty} \frac{\Gamma_n(f)}{\varphi(n)} > 0. \end{cases} \tag{1}$$

For some related questions on the asymptotics of diffeomorphisms with fixed points see [4]. Other recent results on the behavior of the growth sequences are in [2], [5].

Proof of the Theorem: We represent the torus as the product $[0, 1) \times [0, 1)$, and define, as in [4],

$$f(x, y) = (\{x + \alpha\}, \{y + cF(x)\}), \quad x, y \in [0, 1),$$

for $\alpha \in \mathbb{R}$, $c \in (0, +\infty)$, and a real analytic 1-periodic function $F\colon \mathbb{R} \to \mathbb{R}$; here $\{\cdot\}$ stands for the fractional part. Then

$$d_x f = \begin{pmatrix} 1 & 0 \\ cF'(x) & 1 \end{pmatrix},$$

and f is an area-preserving real analytic diffeomorphism of the torus.

Define the Weyl sums

$$W(N,x,\alpha) = \sum_{n=0}^{N-1} F'(x+n\alpha). \tag{2}$$

We have

$$d_x f^N = \begin{pmatrix} 1 & 0 \\ cW(N,x,\alpha) & 1 \end{pmatrix}, \quad N \geq 1,$$

and

$$d_x f^{-N} = \begin{pmatrix} 1 & 0 \\ -cW(N,x-N\alpha,\alpha) & 1 \end{pmatrix}, \quad N \geq 1.$$

Therefore, for (1) to hold it suffices that F and α satisfy the following condition:

$$0 < \limsup_{N\to\infty} \max_{0\leq x<1} \frac{W(N,x,\alpha)}{\varphi(N)} < \infty. \tag{3}$$

Up to now our proof has repeated that of Polterovich in [4]. The main difference of our argument is in the way of estimating the Weyl sums (2).

We are going to choose a sequence $\{q_k\}_{k\geq 1}$, $q_1 = 1$,

$$\frac{q_{k+1}}{100q_k} \in \mathbb{N}, \quad k \geq 1, \tag{4}$$

and a sequence $\{r_k\}_{k\geq 1}$,

$$0 < r_k < \exp(-q_k), \quad k \geq 1, \tag{5}$$

and define

$$F(x) = \sum_{k\geq 1} \frac{r_k}{2\pi q_k} \sin[2\pi q_k x], \quad x \in \mathbb{R}.$$

Then F is real analytic and 1-periodic,

$$F'(x) = \sum_{k\geq 1} r_k \cos[2\pi q_k x], \quad x \in \mathbb{R}.$$

For $\alpha \in \mathbb{R}$ denote

$$\Delta_k(N,\alpha) = \sum_{n=0}^{N-1} e^{2\pi i q_k n\alpha}, \tag{6}$$

$$S(N,\alpha) = \sum_{k\geq 1} r_k \operatorname{Re} \Delta_k(N,\alpha),$$

$$T(N,\alpha) = \sum_{k\geq 1} r_k |\Delta_k(N,\alpha)|.$$

Then

$$\Delta_k(N,\alpha)=\frac{1-e^{2\pi i q_k N\alpha}}{1-e^{2\pi i q_k\alpha}},\quad q_k\alpha\in\mathbb{R}\setminus\mathbb{Z},$$

$$\begin{aligned}W(N,x,\alpha)&=\sum_{k\ge1}r_k\operatorname{Re}\left[e^{2\pi i q_k x}\sum_{n=0}^{N-1}e^{2\pi i q_k n\alpha}\right]\\&=\sum_{k\ge1}r_k\operatorname{Re}\left[e^{2\pi i q_k x}\Delta_k(N,\alpha)\right],\end{aligned}$$

and property (3) follows from the inequalities

$$\limsup_{N\to\infty}\frac{S(N,\alpha)}{\varphi(N)}>0,\tag{7}$$

$$\limsup_{N\to\infty}\frac{T(N,\alpha)}{\varphi(N)}<\infty.\tag{8}$$

To get (7) and (8), we should first study the behavior of the sums $\Delta_k(N,\alpha)$. Essentially, if the fractional part of $q_k\alpha$ is of order $1/M$, then $\Delta_k(N,\alpha)$ behaves like N/M for N smaller than M, and is bounded by a constant times M for all N. After that, in an inductive process we approximate φ from below on an infinite sequence of points by a weighted sum of $\Delta_k(N,\alpha)$, with a lacunary sequence q_k and a suitable α.

Our first observation is as follows. Fix $n\ge1$, suppose that the numbers $q_1,\dots,q_{n+1}$ satisfy condition (4), and define

$$k_n=\sum_{1\le s\le n}\frac{q_n}{q_s}\in\mathbb{N}.$$

Claim: *Suppose that β belongs to the interval*

$$\mathcal{A}_n=\left\{\beta\colon\ \frac{q_n}{q_{n+1}}\le q_n\beta-k_n\le\frac{2q_n}{q_{n+1}}\right\}.\tag{9}$$

Then

$$\left|\frac{\Delta_n(N,\beta)}{N}-1\right|\le\frac12,\quad 1\le N\le\frac{q_{n+1}}{100q_n},\tag{10}$$

$$|\Delta_n(N,\beta)|\le\frac{q_{n+1}}{q_n},\quad N\ge1.\tag{11}$$

Proof: Applying the Taylor formula to the function $x\mapsto\exp ix$, we get

$$|e^{ix}-1-ix|\le\frac{|x|^2}{2},\quad x\in\mathbb{R}.$$

Using this inequality and the conditions $\beta \in \mathcal{A}_n$ and $q_n/q_{n+1} \leq 1/100$, we obtain

$$\left|\frac{e^{2\pi i(q_n\beta-k_n)}-1}{2\pi i(q_n\beta-k_n)}-1\right| \leq \left|\frac{(2\pi)^2(q_n\beta-k_n)^2}{4\pi(q_n\beta-k_n)}\right| \leq \frac{1}{10}. \tag{12}$$

Furthermore, if $1 \leq N \leq q_{n+1}/(100q_n)$, then

$$\begin{aligned}\left|\frac{e^{2\pi iN(q_n\beta-k_n)}-1}{2\pi iN(q_n\beta-k_n)}-1\right| &\leq \left|\frac{(2\pi)^2N^2(q_n\beta-k_n)^2}{4\pi N(q_n\beta-k_n)}\right| \\ &= \pi N(q_n\beta-k_n) \leq \frac{1}{10}.\end{aligned}$$

Hence,

$$\left|\frac{\Delta_n(N,\beta)}{N}-1\right| = \left|\frac{1-e^{2\pi iN(q_n\beta-k_n)}}{N(1-e^{2\pi i(q_n\beta-k_n)})}-1\right| \leq \frac{1}{2}, \quad 1 \leq N \leq \frac{q_{n+1}}{100q_n},$$

and (10) is proved.

Next,

$$|1-e^{2\pi iNq_n\beta}| \leq 2, \quad N \geq 1.$$

Therefore,

$$|\Delta_n(N,\beta)| = \left|\frac{1-e^{2\pi iNq_n\beta}}{1-e^{2\pi iq_n\beta}}\right| \leq \frac{2}{|1-e^{2\pi i(q_n\beta-k_n)}|}.$$

Using (12), we get

$$|\Delta_n(N,\beta)| \leq \frac{3}{2\pi(q_n\beta-k_n)} \leq \frac{q_{n+1}}{q_n}, \quad N \geq 1,$$

and (11) is proved. ∎

Now, to obtain (7) and (8), we define q_k, r_k, and α in an inductive process. Without loss of generality we assume that $\varphi(1) \geq 2$. Set $S^0(N,\beta) = T^0(N,\beta) = 0$, $q_1 = 1$, $N_0 = M_0 = 1$, $\mathcal{A}_0 = [1,2]$. On the induction step $p \geq 1$ we start with sequences $\{q_j\}_{1\leq j\leq p}$, $\{r_j\}_{1\leq j<p}$, $\{N_j\}_{0\leq j<p}$, $\{M_j\}_{0\leq j<p}$, and the interval $\mathcal{A}_{p-1}$ (defined by $\{q_j\}_{1\leq j\leq p}$), such that for every $\beta \in \mathcal{A}_{p-1}$, the function S^{p-1},

$$S^{p-1}(N,\beta) = \sum_{1\leq n<p} r_n \operatorname{Re}\Delta_n(N,\beta),$$

satisfies

$$S^{p-1}(N_j,\beta) \geq \frac{1}{100}\varphi(N_j) + 2^{-p}, \quad 1 \leq j < p, \tag{13}$$

and for every $\beta \in \mathcal{A}_{p-1}$ the function T^{p-1},

$$T^{p-1}(N,\beta) = \sum_{1\le n<p} r_n|\Delta_n(N,\beta)|,$$

satisfies

$$T^{p-1}(N,\beta) \le 200\varphi(N) - 2^{-p}, \quad N \ge 1, \tag{14}$$

$$T^{p-1}(N,\beta) \le \frac{1}{100}\varphi(M_{p-1}), \quad N \ge 1. \tag{15}$$

(It is easy to verify that conditions (13)–(15) are fulfilled for $p = 1$.)

By (6), for any β, N,

$$|\Delta_p(N,\beta)| \le N, \tag{16}$$

and we can choose r_p satisfying (5) such that for all $\beta \in \mathcal{A}_{p-1}$,

$$r_p(N + |\Delta_p(N,\beta)|) \le 2^{-p-1}, \quad 1 \le N \le \max(M_{p-1}, N_{p-1}). \tag{17}$$

Since φ is increasing and $\lim_{N\to\infty}\varphi(N)/N = 0$, we can find the smallest natural number N_p such that $\varphi(N_p) \le r_pN_p$. Then

$$r_pN < \varphi(N), \quad N < N_p, \tag{18}$$

$$\varphi(N_p) \le r_pN_p \le \varphi(N_p) + 1, \tag{19}$$

and by (17),

$$N_p > \max(M_{p-1}, N_{p-1}).$$

Set $q_{p+1} = 100q_pN_p$. Then we define $\mathcal{A}_p$ by the formula (9). It is easily seen that $\mathcal{A}_p \subset \mathcal{A}_{p-1}$. In the estimates to follow we assume that $\beta \in \mathcal{A}_p$, and hence, by the Claim, the estimates (10) and (11) hold with $n = p$.

By (10) and (19),

$$r_p \operatorname{Re}\Delta_p(N_p,\beta) \ge \frac{1}{2}r_pN_p \ge \frac{1}{2}\varphi(N_p). \tag{20}$$

By (16) and (18),

$$r_p|\Delta_p(N,\beta)| \le r_pN \le \varphi(N), \quad N < N_p, \tag{21}$$

and by (11) and (19),

$$r_p|\Delta_p(N,\beta)| \le 100r_pN_p \le 100(\varphi(N_p) + 1), \quad N \ge N_p. \tag{22}$$

Using that φ is increasing, we conclude that

$$r_p|\Delta_p(N,\beta)| \leq 100(\varphi(N)+1), \quad N \geq 1. \tag{23}$$

Now, by (13) and (17),

$$\begin{aligned} S^p(N_j,\beta) &\geq S^{p-1}(N_j,\beta) - r_p|\Delta_p(N_j,\beta)| \\ &\geq \frac{1}{100}\varphi(N_j) + 2^{-p-1}, \quad 1 \leq j < p, \end{aligned}$$

and by (15) and (20),

$$S^p(N_p,\beta) \geq r_p \operatorname{Re}\Delta_p(N_p,\beta) - T^{p-1}(N_p,\beta) \geq \frac{1}{100}\varphi(N_p) + 2^{-p-1}.$$

Thus,

$$S^p(N_j,\beta) \geq \frac{1}{100}\varphi(N_j) + 2^{-p-1}, \quad 1 \leq j \leq p.$$

Furthermore, by (14) and (17),

$$\begin{aligned} T^p(N,\beta) &= T^{p-1}(N,\beta) + r_p|\Delta_p(N,\beta)| \leq 200\varphi(N) - 2^{-p} + 2^{-p-1} \\ &= 200\varphi(N) - 2^{-p-1}, \quad 1 \leq N \leq M_{p-1}, \end{aligned}$$

and by (15) and (23),

$$\begin{aligned} T^p(N,\beta) &= T^{p-1}(N,\beta) + r_p|\Delta_p(N,\beta)| \\ &\leq \frac{1}{100}\varphi(M_{p-1}) + 100(\varphi(N)+1), \quad N \geq 1. \end{aligned}$$

Since φ is increasing, we conclude that

$$T^p(N,\beta) \leq 200\varphi(N) - 1, \quad N > M_{p-1},$$

and hence

$$T^p(N,\beta) \leq 200\varphi(N) - 2^{-p-1}, \quad N \geq 1.$$

Finally, by (15), (21), (22), and by the condition that φ is unbounded, there exists M_p such that

$$T^p(N,\beta) \leq \frac{1}{100}\varphi(M_p), \quad N \geq 1.$$

Thus, the inequalities (13)–(15) hold with p replaced by $p+1$. This completes the induction step.

The intervals $\mathcal{A}_n$ constitute a nested family,

$$\bigcap_{n\geq 0} \mathcal{A}_n = \{\alpha\},$$
$$\alpha = \sum_{k\geq 1} \frac{1}{q_k},$$

and all the inequalities in the induction process are valid with $\beta = \alpha$.

We have

$$S(N,\alpha) = \lim_{p\to\infty} S^p(N,\alpha), \quad T(N,\alpha) = \lim_{p\to\infty} T^p(N,\alpha).$$

Then the properties (13) and (14) imply that

$$\frac{S(N_j,\alpha)}{\varphi(N_j)} \geq \frac{1}{100}, \quad j \geq 1,$$
$$\frac{T(N,\alpha)}{\varphi(N)} \leq 200, \quad N \geq 1,$$

and (7) and (8) follow. The theorem is proved. ∎

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